FUZZY STABILITY OF ADDITIVE–QUADRATIC FUNCTIONAL EQUATIONS

M. ESHAGHI GORDJI, N.GHOBADIPOUR AND J. M. RASSIAS

Abstract. In this paper we investigate the generalized Hyers- Ulam stability of the functional equation

$$f(2x+y)+f(2x-y)=f(x+y)+f(x-y)+2f(2x)-2f(x)$$

in fuzzy Banach spaces.

1. Introduction and preliminaries

In 1984, Katsaras [17] defined a fuzzy norm on a linear space and at the same year Wu and Fang [32] also introduced a notion of fuzzy normed space and gave the generalization of the Kolmogoroff normalized theorem for fuzzy topological linear space. In [6], Biswas defined and studied fuzzy inner product spaces in linear space. Since then some mathematicians have defined fuzzy metrics and norms on a linear space from various points of view [5, 10, 19, 28, 31]. In 1994, Cheng and Mordeson introduced a definition of fuzzy norm on a linear space in such a manner that the corresponding induced fuzzy metric is of Kramosil and Michalek type [18]. In 2003, Bag and Samanta [5] modified the definition of Cheng and Mordeson [7] by removing a regular condition. They also established a decomposition theorem of a fuzzy norm into a family of crisp norms and investigated some properties of fuzzy norms (see [4]). Following [3], we give the employing notion of a fuzzy norm.

Let X be a real linear space. A function $N: X\times\mathbb{R}\longrightarrow[0,1]$(the so-called fuzzy subset) is said to be a fuzzy norm on X if for all $x,y\in X$ and all $a,b\in\mathbb{R}$:

(N_1) $N(x,a)=0$ for $a\leq 0$;

(N_2) $x=0$ if and only if $N(x,a)=1$ for all $a>0$;

(N_3) $N(ax,b)=N(x,\frac{b}{|a|})$ if $a\neq 0$;

(N_4) $N(x+y,a+b)\geq min\{N(x,a),N(y,b)\}$;

(N_5) $N(x,.)$ is non-decreasing function on $\mathbb{R}$ and $\lim_{a\to\infty}N(x,a)=1$;

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(N_6) For $x \neq 0$, $N(x, .)$ is (upper semi) continuous on $\mathbb{R}$.
The pair (X, N) is called a fuzzy normed linear space. One may regard $N(x, a)$ as the truth value of the statement the norm of x is less than or equal to the real number $a^{'}$.

Example 1.1. Let $(X, \|.\|)$ be a normed linear space. Then

$$N(x,a) = \begin{cases} \frac{a}{a+\|x\|}, & a > 0\ , x \in X, \\ 0, & a \leq 0, x \in X \end{cases}$$

is a fuzzy norm on X.

Definition 1.2. Let (X, N) be a fuzzy normed linear space. Let x_n be a sequence in X. Then x_n is said to be convergent if there exists $x \in X$ such that $\lim_{n\to\infty} N(x_n - x, a) = 1$ for all $a > 0$. In that case, x is called the limit of the sequence x_n and we denote it by $N - lim_{n\to\infty}\ x_n = x$.

Definition 1.3. A sequence x_n in X is called Cauchy if for each $\epsilon > 0$ and each $a > 0$ there exists n_0 such that for all $n \geq n_0$ and all $p > 0$, we have $N(x_{n+p} - x_n, a) > 1$ - ϵ.

It is known that every convergent sequence in fuzzy normed space is Cauchy. If each Cauchy sequence is convergent, then the fuzzy norm is said to be complete and the fuzzy normed space is called a fuzzy Banach space.
The study of the stability problem of functional equations was introduced by Ulam [30] Let $(G_1, .)$ be a group and let $(G_2, *)$ be a metric group with the metric $d(.,.)$. Given $\epsilon > 0$, does there exist a $\delta > 0$, such that if a mapping $h : G_1 \longrightarrow G_2$ satisfies the inequality $d(h(x.y), h(x) * h(y)) < \delta$ for all $x, y \in G_1$, then there exists a homomorphism $H : G_1 \longrightarrow G_2$ with $d(h(x), H(x)) < \epsilon$ for all $x \in G_1$? In other words, under what condition does there exist a homomorphism near an approximate homomorphism? The concept of stability for functional equations arises when we replace the functional equation by an inequality which acts as a perturbation of the equation. In 1941, D. H. Hyers [13] gave the first affirmative answer to the question of Ulam for Banach spaces E and E'. Let $f : E \longrightarrow E'$ be a mapping between Banach spaces such that

$$\|f(x + y) - f(x) - f(y)\| \leq \delta$$

for all $x, y \in E$, and for some $\delta > 0$. Then there exists a unique additive mapping $T : E \longrightarrow E'$ such that

$$\|f(x) - T(x)\| \leq \delta$$

for all $x \in E$. Now assume that E and E' are real normed spaces with E' complete, $f : E \to E'$ is a mapping such that the mapping $t \rightarrowtail f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in E$, and that there exist $\delta \geq 0$ and $p \neq 1$ such that

$$\|f(x+y) - f(x) - f(y)\| \leq \delta(\|x\|^p + \|y\|^p)$$

for all $x, y \in E$. Then there exists a unique linear map $T : E \to E'$ such that

$$\|f(x) - T(x)\| \leq \frac{2\delta\|x\|^p}{|2^p - 2|}$$

for all $x \in E$. (see [27]).
On the other hand J. M. Rassias [23, 24, 25, 26] generalized the Hyers stability result by presenting a weaker condition controlled by a product of different powers of norms. According to J. M. Rassias Theorem:

Theorem 1.4. *If it is assumed that there exist constants $\Theta \geq 0$ and $p_1, p_2 \in \mathbb{R}$ such that $p = p_1 + p_2 \neq 1$, and $f : E \to E'$ is a map from a norm space E into a Banach space E' such that the inequality*

$$\|f(x+y) - f(x) - f(y)\| \leq \epsilon\|x\|^{p_1}\|y\|^{p_2}$$

for all $x, y \in E$, then there exists a unique additive mapping $T : E \to E'$ such that

$$\|f(x) - T(x)\| \leq \frac{\Theta}{2 - 2^p}\|x\|^p,$$

for all $x \in E$. If in addition for every $x \in E$, $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed x, then T is linear.

Quadratic functional equation was used to characterize inner product spaces [1, 2, 14]. Several other functional equations were also used to characterize inner product spaces. A square norm on an inner product space satisfies the important parallelogram equality

$$\|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2).$$

The functional equation

$$f(x+y) + f(x-y) = 2f(x) + 2f(y) \tag{1.1}$$

is related to a symmetric bi-additive function [1, 16]. It is natural that each equation is called a quadratic functional equation. In particular, every solution of the quadratic equation (1.1) is said to be a quadratic function. It is well known that a function f between real vector spaces is quadratic if and only if there exists a unique symmetric bi-additive

function B such that $f(x) = B(x,x)$ for all x (see [1, 16]). The bi-additive function B is given by

$$B(x,y) = \frac{1}{4}(f(x+y) - f(x-y)). \tag{1.2}$$

A Hyers-Ulam stability problem for the quadratic functional equation (1.1) was proved by Skof for functions $f : E_1 \longrightarrow E_2$ where E_1 is a normed space and E_2 is a Banach space (see [29]). Cholewa [8] noticed that the theorem of Skof is still true if the relevant domain E_1 is replaced by an Abelian group. In the paper [9], Czerwik proved the generalized Hyers-Ulam stability of the quadratic functional equation (1.1). Grabiec [12] has generalized these results mentioned above. Jun and Lee [15] proved the generalized Hyers-Ulam stability of the pexiderized quadratic equation (1.1).

A. Najati and M.B. Moghimi [22], have obtained the generalized Hyers- Ulam stability for a functional equation deriving from quadratic and additive functions in quasi-Banach spaces.

In this paper, we deal with the the following functional equation deriving from quadratic and additive functions:

$$f(2x+y) + f(2x-y) = f(x+y) + f(x-y) + 2f(2x) - 2f(x). \tag{1.3}$$

It is easy to see that the function $f(x) = ax^2+bx+c$ is a solution of the functional equation (1.3). The main purpose of this paper is to establish some versions of the generalized Hyers-Ulam stability for the function equation (1.3) in fuzzy normed linear spaces.

2. Main result

Throughout this section, assume that X, (Y, N) and (Z, N') are linear space, fuzzy normed space and fuzzy Banach space, respectively. We start our works with a fuzzy generalized Hyers- Ulam type theorem for the functional equation (1.3).

Theorem 2.1. *Let $\varphi_1 : X \times X \to Z$ be a function such that for some $0 < \alpha < 4$*

$$N'(\varphi_1(\frac{2x}{3}, 2y), a) \geq N'(\alpha\varphi_1(\frac{x}{3}, y), a) \tag{2.1}$$

for all $x \in X$, $y \in \{0, \frac{x}{3}, \frac{4x}{3}, \frac{-2x}{3}, x\}$ and $a > 0$, and $\lim_{n\to\infty} N'(\varphi_1(2^n x, 2^n y), 4^n a) = 1$ for all $x, y \in X$ and $a > 0$. Let $f : X \to Y$ be an even function with $f(0) = 0$ satisfying

$$N(f(2x+y)+f(2x-y)-f(x+y)-f(x-y)-2f(2x)+2f(x), a) \geq N'(\varphi_1(x,y), a) \tag{2.2}$$

for all $a > 0$ and all $x, y \in X$. Then there exists a unique quadratic mapping $Q : X \to Y$ such that

$$N(Q(x) - f(x), a) \geq N_1''(x, \frac{a(4-\alpha)}{6}) \tag{2.3}$$

for all $x \in X$ and all $a > 0$, where

$$N_1''(x,a) := min\{N'(\varphi_1(\frac{x}{3},\frac{x}{3}),a), N'(\varphi_1(\frac{x}{3},x),a), \\ N'(\varphi_1(\frac{x}{3},\frac{4x}{3}),a), N'(\varphi_1(\frac{x}{3},\frac{-2x}{3}),a), N'(\varphi_1(\frac{x}{3},0),a)\}.$$

Proof. By replacing y by $x+y$ in (2.2), we get

$$N(f(3x+y)+f(x-y)-f(2x+y)-f(y)-2f(2x)+2f(x),a) \geq N'(\varphi_1(x,x+y),a) \quad (2.4)$$

for all $x,y \in X$ and $a > 0$. Replacing y by $-y$ in (2.4), we get

$$N(f(3x-y)+f(x+y)-f(2x-y)-f(y)-2f(2x)+2f(x),a) \\ \geq N'(\varphi_1(x,x-y),a) \quad (2.5)$$

for all $x,y \in X$ and $a > 0$. It follows from (2.2), (2.4) and (2.5),

$$N(f(3x+y)+f(3x-y)-2f(y)-6f(2x)+6f(x),3a) \\ \geq min\{N'(\varphi_1(x,y),a), N'(\varphi_1(x,x+y),a), N'(\varphi_1(x,x-y),a)\} \quad (2.6)$$

for all $x,y \in X$ and $a > 0$. Letting $y = 0$ in (2.6), we get inequality

$$N(2f(3x)-6f(2x)+6f(x),3a) \geq min\{N'(\varphi_1(x,x),a), N'(\varphi_1(x,0),a)\} \quad (2.7)$$

for all $x,y \in X$ and $a > 0$. Putting $y = 3x$ in (2.6), we get

$$N(f(6x)-2f(3x)-6f(2x)+6f(x),3a) \\ \geq min\{N'(\varphi_1(x,3x),a), N'(\varphi_1(x,4x),a), N'(\varphi_1(x,-2x),a)\} \quad (2.8)$$

$x,y \in X$ and $a > 0$. It follows from (2.7) and (N_3),

$$N(-2f(3x)+6f(2x)-6f(x),3a) \geq min\{N'(\varphi_1(x,x),a), N'(\varphi_1(x,0),a)\} \quad (2.9)$$

for all $x,y \in X$ and $a > 0$. Therefore we to obtain from (2.8) and (2.9) the inequality

$$N(f(6x)-4f(3x),6a) \geq min\{N'(\varphi_1(x,x),a), N'(\varphi_1(x,3x),a), \\ N'(\varphi_1(x,4x),a), N'(\varphi_1(x,-2x),a), N'(\varphi_1(x,0),a)\} \quad (2.10)$$

for all $x,y \in X$ and $a > 0$. If we replace x by $\frac{x}{3}$ in (2.10) for all $x \in X$ and $a > 0$, then we get then

$$N(f(2x)-4f(x),6a) \geq N_1''(x,a) \quad (2.11)$$

for all $x \in X$ and $a > 0$. Thus

$$N(\frac{f(2x)}{4}-f(x),\frac{3a}{2}) \geq N_1''(x,a) \quad (2.12)$$

for all $x \in X$ and $a > 0$. Replacing x by $2^n x$ in (2.12), we get

$$N(\frac{f(2^{n+1}x)}{4} - f(2^n x), \frac{3a}{2}) \geq N_1''(2^n x, a) \tag{2.13}$$

for all $x \in X$ and $a > 0$. Using (2.1) we get

$$N(\frac{f(2^{n+1}x)}{4} - f(2^n x), \frac{3a}{2}) \geq N_1''(x, \frac{a}{\alpha^n}) \tag{2.14}$$

for all $x \in X$ and $a > 0$. Replacing a by $\alpha^n a$ we see that

$$N(\frac{f(2^{n+1}x)}{4^{n+1}} - \frac{f(2^n x)}{4^n}, \frac{3a\alpha^n}{2(4^n)}) \geq N_1''(x, a) \tag{2.15}$$

for all $x \in X$ and $a > 0$. It follows from $\frac{f(2^n x)}{4^n} - f(x) = \sum_{i=0}^{n-1} \frac{f(2^{i+1}x)}{4^{i+1}} - \frac{f(2^i x)}{4^i}$ and (2.15) that

$$N(\frac{f(2^n x)}{4^n} - f(x), \sum_{i=0}^{n-1} \frac{3a\alpha^i}{2(4^i)}) \geq min \bigcup_{i=0}^{n-1} \{N(\frac{f(2^{i+1}x)}{4^{i+1}} - \frac{f(2^i x)}{4^i}, \frac{3a\alpha^i}{2(4^i)})\} \geq N_1''(x, a) \tag{2.16}$$

for all $x \in X$ and $a > 0$. Replacing x with $2^m x$ in (2.16) we observe that

$$N(\frac{f(2^{n+m}x)}{4^{n+m}} - \frac{f(2^m x)}{4^m}, \sum_{i=0}^{n-1} \frac{3a\alpha^i}{2(4^{i+m})}) \geq N_1''(2^m x, a) \geq N_1''(x, \frac{a}{\alpha^m}),$$

whence

$$N(\frac{f(2^{n+m}x)}{4^{n+m}} - \frac{f(2^m x)}{4^m}, \sum_{i=m}^{n+m-1} \frac{3a\alpha^i}{2(4^i)}) \geq N_1''(x, a)$$

for all $x \in X$, $a > 0$ and $m, n \geq 0$.
Hence

$$N(\frac{f(2^{n+m}x)}{4^{n+m}} - \frac{f(2^m x)}{4^m}, a)) \geq N_1''(x, \frac{a}{\sum_{i=m}^{n+m-1} \frac{3\alpha^i}{2(4^i)}}) \tag{2.17}$$

for all $x \in X$, $a > 0$ and $m, n \geq 0$. Since $0 < \alpha < 4$ and $\sum_{i=0}^{\infty} (\frac{\alpha}{4})^i < \infty$ the Cauchy criterion for convergence and (N_5) imply that $\{\frac{f(2^n x)}{4^n}\}$ is a Cauchy sequence in (Y, N). Since (Y, N) is a fuzzy Banach space, this sequence converges to some point $Q(x) \in Y$. So one can define the mapping $Q : X \to Y$ by $Q(x) := N - \lim_{n\to\infty} \frac{f(2^n x)}{4^n}$ for all $x \in X$. Letting $m = 0$ in (2.17), we get

$$N(\frac{f(2^n x)}{4^n} - f(x), a) \geq N_1''(x, \frac{a}{\sum_{i=0}^{n-1} \frac{3\alpha^i}{2(4^i)}}) \tag{2.18}$$

for all $x \in X$ and $a > 0$. Taking the limit as $n \to \infty$ and using (N_6) we get

$$N(Q(x) - f(x), a) \geq N_1''(x, \frac{a(4-\alpha)}{6})$$

for all $x \in X$ and $a > 0$. Now, we claim that Q is quadratic. Replace x, y by $2^n x, 2^n y$, respectively in (2.2) to get

$$\begin{aligned} N(&\frac{f(2^n(2x+y))}{4^n} + \frac{f(2^n(2x-y))}{4^n} - \frac{f(2^n(x+y))}{4^n} \\ &- \frac{f(2^n(x-y))}{4^n} - \frac{2f(2^n(2x))}{4^n} + \frac{2f(2^n x)}{4^n}, a) \\ &\geq N^{'}(\varphi_1(2^n x, 2^n y), 4^n a) \end{aligned}$$

for all $x, y \in X$ and $a > 0$. Since $\lim_{n\to\infty} N^{'}(\varphi_1(2^n x, 2^n y), 4^n a) = 1$ and $Q(0) = 0$, then by Lemma 2.1 of [22] we get that the mapping $Q : X \to Y$ is quadratic.
To prove the uniqueness of Q, let $Q^{'} : X \to Y$ be another quadratic mapping satisfying (2.3). Fix $x \in X$. Clearly $Q(2^n x) = 4^n Q(x)$ and $Q^{'}(2^n x) = 4^n Q^{'}(x)$ for all $n \in \mathbb{N}$. It follows from (2.3) that

$$\begin{aligned} N(Q(x) - Q^{'}(x), a) &= N(\frac{Q(2^n x)}{4^n} - \frac{Q^{'}(2^n x)}{4^n}, a) \\ &\geq min\{N(\frac{Q(2^n x)}{4^n} - \frac{f(2^n x)}{4^n}, \frac{a}{2}), N(\frac{f(2^n x)}{4^n} - \frac{Q^{'}(2^n x)}{4^n}, \frac{a}{2})\} \\ &\geq N_1^{''}(2^n x, \frac{a(4-\alpha)(4^n)}{12}) \geq N_1^{''}(x, \frac{a(4-\alpha)(4^n)}{12\alpha^n}) \end{aligned}$$

for all $x \in X$ and $a > 0$.
Since $\lim_{n\to\infty} \frac{a(4-\alpha)(4^n)}{12\alpha^n} = \infty$, we obtain $\lim_{n\to\infty} N_1^{''}(x, \frac{a(4-\alpha)(4^n)}{12\alpha^n}) = 1$. Therefore, $N(Q(x) - Q^{'}(x), a) = 1$ for all $x \in X$ and all $a > 0$, whence $Q(x) = Q^{'}(x)$. □

Theorem 2.2. *Let $\varphi_2 : X \times X \to Z$ be a function such that for some $\alpha > 4$*

$$N^{'}(\varphi_2(\frac{x}{2(3)}, \frac{y}{2}), a) \geq N^{'}(\varphi_2(\frac{x}{2(3)}, y), \alpha a)$$

for all $x \in X$, $y \in \{0, \frac{x}{3}, \frac{4x}{3}, \frac{-2x}{3}, x\}$ and $a > 0$, and $\lim_{n\to\infty} N^{'}(4^n \varphi_2(2^{-n}x, 2^{-n}y), a) = 1$ for all $x, y \in X$ and $a > 0$. Let $f : X \to Y$ be an even function with $f(0) = 0$ satisfies (2.2) for all $a > 0$ and all $x, y \in X$. Then there exists a unique quadratic mapping $Q : X \to Y$ such that

$$N(Q(x) - f(x), a) \geq N_2^{''}(x, \frac{a(\alpha-4)}{6})$$

for all $x \in X$ and all $a > 0$, where

$$\begin{aligned} N_2^{''}(x, a) := min\{&N^{'}(\varphi_2(\frac{x}{3}, \frac{x}{3}), a), N^{'}(\varphi_2(\frac{x}{3}, x), a), \\ &N^{'}(\varphi_2(\frac{x}{3}, \frac{4x}{3}), a), N^{'}(\varphi_2(\frac{x}{3}, \frac{-2x}{3}), a), N^{'}(\varphi_2(\frac{x}{3}, 0), a)\}. \end{aligned}$$

Proof. The techniques are completely similar to those techniques of Theorem 2.1. Hence we present a sketch of proof. If we replace x by $\frac{x}{2^{n+1}}$ in (2.11), then we have

$$N(4f(\frac{x}{2^{n+1}})-f(\frac{x}{2^n}),6a)\geq N_2^{''}(\frac{x}{2^{n+1}},a)$$

whence

$$N(4^{n+1}f(\frac{x}{2^{n+1}})-4^nf(\frac{x}{2^n}),6(4^n)a)\geq N_2^{''}(\frac{x}{2^{n+1}},a) \tag{2.19}$$

for all $x\in X$ and $a>0$. One can deduce

$$N(4^{n+m}f(\frac{x}{2^{n+m}})-4^mf(\frac{x}{2^m}),a)\geq N_2^{''}(x,\frac{a}{\sum_{i=1}^{n+m}\frac{6}{\alpha}(\frac{4}{\alpha})^i}) \tag{2.20}$$

for all $x\in X, n\geq 0, m\geq 0$ and $a>0$. Hence, we conclude that $\{4^nf(\frac{x}{2^n})\}$ is a Cauchy sequence in the fuzzy Banach space (Y,N). Therefore, there is a function $Q:X\to Y$ defined by $Q(x):=N-\lim_{n\to\infty}4^nf(\frac{x}{2^n})$. Employing (2.20) with $m=0$ we obtain

$$N(Q(x)-f(x),a)\geq N_2^{''}(x,\frac{a(4-\alpha)}{6})$$

for all $x\in X$ and all $a>0$. □

Theorem 2.3. *Let $\varphi_3:X\times X\to Z$ be a function such that for some $0<\alpha<2$*

$$N^{'}(\varphi_3(2(\frac{x}{2}),2y))\geq N^{'}(\alpha\varphi_3((\frac{x}{2}),y)) \tag{2.21}$$

for all $x\in X$, $y\in\{x,\frac{x}{2},\frac{3x}{2},2x\}$ and $a>0$, and $\lim_{n\to\infty}N^{'}(\varphi_3(2^nx,2^ny),2^na)=1$ for all $x,y\in X$ and $a>0$. Let $f:X\to Y$ be an odd function satisfying (2.2) for all $a>0$ and all $x,y\in X$. Then there exists a unique additive mapping $A:X\to Y$ such that

$$N(A(x)-f(x),a)\geq N_3^{''}(x,\frac{a(2-\alpha)}{4}) \tag{2.22}$$

for all $x\in X$ and all $a>0$, where

$$N_3^{''}(x,a):=min\{N^{'}(\varphi_3(x,x),a),N^{'}(\varphi_3(\frac{x}{2}),a),$$
$$N^{'}(\varphi_3(\frac{x}{2},2x),a),N^{'}(\varphi_3(\frac{x}{2},\frac{3x}{2}),a)\}.$$

Proof. Replacing y by x in (2.2), we get

$$N(f(3x)-3f(2x)+3f(x),a)\geq N^{'}(\varphi_3(x,x),a) \tag{2.23}$$

for all $x\in X$ and all $a>0$. Replacing y by $3x$ in (2.2), we get

$$N(f(5x)-f(4x)-f(2x)+f(x),a)\geq N^{'}(\varphi_3(x,3x),a) \tag{2.24}$$

for all $x\in X$ and all $a>0$. Putting $y=4x$ in (2.2) we obtain

$$N(f(6x)-f(5x)+f(3x)-3f(2x)+2f(x),a)\geq N^{'}(\varphi_3(x,4x),a) \tag{2.25}$$

for all $x \in X$ and all $a > 0$. It follows from (2.23), (2.24) and (2.25),

$$N(f(6x) - f(4x) - f(2x), 3a) \geq min\{N^{'}(\varphi_3(x, x), a), N^{'}(\varphi_3(x, 3x), a), \\ N^{'}(\varphi_3(x, 4x), a)\} \qquad (2.26)$$

for all $x \in X$ and all $a > 0$. If we replace x by $\frac{x}{2}$ in (2.26), then

$$N(f(3x) - f(2x) - f(x), 3a) \geq min\{N^{'}(\varphi_3(\frac{x}{2}, \frac{x}{2}), a), N^{'}(\varphi_3(\frac{x}{2}, 2x), a), \\ N^{'}(\varphi_3(\frac{x}{2}, \frac{3x}{2}), a) \qquad (2.27)$$

for all $x \in X$ and $a > 0$. It follows from (2.23) and (2.27),

$$N(\frac{f(2x)}{2} - f(x), 2a) \geq N_3^{''}(x, a) \qquad (2.28)$$

for all $x \in X$ and all $a > 0$. Replacing x by $2^n x$ in (2.28), we get

$$N(\frac{f(2^{n+1}x)}{2} - f(2^n x), 2a) \geq N_3^{''}(x, a)(2^n x, a) \qquad (2.29)$$

for all $x \in X$ and all $a > 0$. Using (2.21) we get

$$N(\frac{f(2^{n+1}x)}{2} - f(2^n x), 2a) \geq N_3^{''}(x, \frac{a}{\alpha^n}) \qquad (2.30)$$

for all $x \in X$ and all $a > 0$. Replacing a by $\alpha^n a$ we see that

$$N(\frac{f(2^{n+1}x)}{2^{n+1}} - \frac{f(2^n x)}{2^n}, \frac{2a\alpha^n}{2^n}) \geq N_3^{''}(x, a) \qquad (2.31)$$

for all $x \in X$ and all $a > 0$. It follows from $\frac{f(2^n x)}{2^n} - f(x) = \sum_{i=0}^{n-1} \frac{f(2^{i+1}x)}{2^{i+1}} - \frac{f(2^i x)}{2^i}$ and (2.31) that

$$N(\frac{f(2^n x)}{2^n} - f(x), \sum_{i=0}^{n-1} \frac{2a\alpha^i}{2^i}) \geq min \bigcup_{i=0}^{n-1} \{N(\frac{f(2^{i+1}x)}{2^{i+1}} - \frac{f(2^i x)}{2^i}, \frac{2a\alpha^i}{2^i})\} \geq N_3^{''}(x, a) \quad (2.32)$$

for all $x \in X$ and all $a > 0$. By replacing x with $2^m x$ in (2.32) we observe that

$$N(\frac{f(2^{n+m}x)}{2^{n+m}} - \frac{f(2^m x)}{2^m}, \sum_{i=0}^{n-1} \frac{2a\alpha^i}{2^{i+m}}) \geq N_3^{''}(2^m x, a) \geq N_3^{''}(x, \frac{a}{\alpha^m}),$$

whence

$$N(\frac{f(2^{n+m}x)}{2^{n+m}} - \frac{f(2^m x)}{2^m}, \sum_{i=m}^{n+m-1} \frac{2a\alpha^i}{2^i}) \geq N_3^{''}(x, a)$$

for all $x \in X$, $a > 0$ and $m, n \geq 0$.
Hence

$$N(\frac{f(2^{n+m}x)}{2^{n+m}} - \frac{f(2^m x)}{2^m}, a)) \geq N_3^{''}(x, \frac{a}{\sum_{i=m}^{n+m-1} \frac{2\alpha^i}{2^i}}) \qquad (2.33)$$

for all $x \in X$, $a > 0$ and $m, n \geq 0$. Since $0 < \alpha < 2$ and $\sum_{i=0}^{\infty}(\frac{\alpha}{2})^i < \infty$ the Cauchy criterion for convergence and (N_5) show that $\{\frac{f(2^n x)}{2^n}\}$ is a Cauchy sequence in (Y, N). Since (Y, N) is a fuzzy Banach space, this sequence converges to some point $A(x) \in Y$. So one can define the mapping $A : X \to Y$ by $A(x) := N - \lim_{n\to\infty} \frac{f(2^n x)}{2^n}$ for all $x \in X$. Letting $m = 0$ in (2.33), we get

$$N(\frac{f(2^n x)}{2^n} - f(x), a) \geq N_3^{''}(x, \frac{a}{\sum_{i=0}^{n-1} \frac{2\alpha^i}{2^i}}) \tag{2.34}$$

for all $x \in X$, and $a > 0$. Taking the limit as $n \to \infty$ and using (N_6) we get

$$N(A(x) - f(x), a) \geq N_3^{''}(x, a)$$

for all $x \in X$ and all $a > 0$. Now, we show that A is additive. Replace x, y by $2^n x, 2^n y$, respectively in (2.2) to get

$$\begin{aligned} N(&\frac{f(2^n(2x+y))}{2^n} + \frac{f(2^n(2x-y))}{2^n} - \frac{f(2^n(x+y))}{2^n} \\ &- \frac{f(2^n(x-y))}{2^n} - \frac{2f(2^n(2x))}{2^n} + \frac{2f(2^n x)}{2^n}, a) \\ &\geq N^{'}(\varphi_3(2^n x, 2^n y), 2^n a) \end{aligned}$$

for all $x, y \in X$ and $a > 0$. Since $\lim_{n\to\infty} N^{'}(\varphi_3(2^n x, 2^n y), 2^n a) = 1$, then by Lemma 2.2 of [22] we get that the mapping $A : X \to Y$ is additive.

To prove the uniqueness of A, let $A^{'} : X \to Y$ be another additive mapping satisfying (2.22). Fix $x \in X$. Clearly $A(2^n x) = 2^n A(x)$ and $A^{'}(2^n x) = 2^n A^{'}(x)$ for all $n \in \mathbb{N}$. It follows from (2.22) that

$$\begin{aligned} N(A(x) - A^{'}(x), a) &= N(\frac{A(2^n x)}{2^n} - \frac{A^{'}(2^n x)}{2^n}, a) \\ &\geq min\{N(\frac{A(2^n x)}{2^n} - \frac{f(2^n x)}{2^n}, \frac{a}{2}), N(\frac{f(2^n x)}{2^n} - \frac{A^{'}(2^n x)}{2^n}, \frac{a}{2})\} \\ &\geq N_3^{''}(2^n x, \frac{2^n a(2-\alpha)}{8}) \geq N_3^{''}(x, \frac{2^n a(2-\alpha)}{8\alpha^n}) \end{aligned}$$

for all $x \in X$ and all $a > 0$.

Since $\lim_{n\to\infty} \frac{a(2^n)(2-\alpha)}{8\alpha^n} = \infty$, we obtain $\lim_{n\to\infty} N_3^{''}(x, \frac{2^n a(2-\alpha)}{8\alpha^n}) = 1$. Therefore, $N(A(x) - A^{'}(x), a) = 1$ for all $a > 0$, whence $A(x) = A^{'}(x)$. □

Theorem 2.4. *Let $\varphi_4 : X \times X \to Z$ be a function such that for some $\alpha > 2$*

$$N^{'}(\varphi_4(\frac{1}{2}(\frac{x}{2}), \frac{y}{2}), a) \geq N^{'}(\varphi_4((\frac{x}{2}), y), \alpha a)$$

for all $x \in X$, $y \in \{x, \frac{x}{2}, \frac{3x}{2}, 2x\}$ *and* $a > 0$, *and* $\lim_{n\to\infty} N^{'}(2^n\varphi_4(2^{-n}x, 2^{-n}y), a) = 1$ *for all* $x, y \in X$ *and* $a > 0$. *Let* $f : X \to Y$ *be an odd function satisfying* (2.2) *for all* $a > 0$ *and all* $x, y \in X$. *Then there exists a unique additive mapping* $A : X \to Y$ *such that*

$$N(A(x) - f(x), a) \geq N_4^{''}(x, \frac{a(\alpha - 2)}{4})$$

for all $x \in X$ *and all* $a > 0$, *where*

$$N_4^{''}(x, a) := min\{N^{'}(\varphi_4(x, x), a), N^{'}(\varphi_4(\frac{x}{2}), a), N^{'}(\varphi_4(\frac{x}{2}, 2x), a), N^{'}(\varphi_4(\frac{x}{2}, \frac{3x}{2}), a)\}.$$

Proof. If we replace x by $\frac{x}{2^{n+1}}$ in (2.28), then we have

$$N(f(\frac{x}{2^n}) - 2f(\frac{x}{2^{n+1}}), a) \geq N_4^{''}(\frac{x}{2^{n+1}}, a)$$

whence

$$N(2^n f(\frac{x}{2^n}) - 2^{n+1}f(\frac{x}{2^{n+1}}), (2^n)a) \geq N_4^{''}(\frac{x}{2^{n+1}}, a)$$

for all $x \in X$ and $a > 0$. One can deduce

$$N(2^m f(\frac{x}{2^m}) - 2^{n+m}f(\frac{x}{2^{n+m}}), a) \geq N_4^{''}(x, \frac{a}{\sum_{i=1}^{n+m} \frac{1}{\alpha}(\frac{2}{\alpha})^i}) \tag{2.35}$$

for all $x \in X, n \geq 0, m \geq 0$ and $a > 0$. Hence, we conclude that $\{2^n f(\frac{x}{2^n})\}$ is a Cauchy sequence in the fuzzy Banach space (Y, N). Therefore, there is a function $A : X \to Y$ defined by $A(x) := N - \lim_{n\to\infty} 2^n f(\frac{x}{2^n})$. Employing (2.35) with $m = 0$ we obtain

$$N(A(x) - f(x), a) \geq N_4^{''}(x, a)$$

for all $x \in X$ and all $a > 0$.
The rest of the proof is similar to the proof of theorem 2.3. □

We now prove our main theorem in paper.

Theorem 2.5. *Let* $\varphi : X \times X \to Z$ *be a function such that for some* $0 < \alpha < 2$

$$N^{'}(\varphi(2(\frac{x}{2}), 2y), a) \geq N^{'}(\alpha\varphi(\frac{x}{2}, y), a)$$

for all $x \in X$, $y \in \beta\{0, x, \frac{x}{2}, \frac{4x}{3}, \frac{-2x}{3}, \frac{x}{3}, \frac{3x}{2}, 2x\}$ *and* $a > 0$, *and* $\lim_{n\to\infty} N^{'}(\varphi(2^n x, 2^n y), 2^n a) = 1$ *for all* $x, y \in X$ *and* $a > 0$. *Let* $f : X \to Y$ *with* $f(0) = 0$ *be a function satisfying* (2.2) *for all* $a > 0$ *and all* $x, y \in X$. *Then there exist a unique quadratic mapping* $Q : X \to Y$ *and a unique additive mapping* $A : X \to Y$ *satisfying* (1.3) *and*

$$N(Q(x) - A(x) - f(x), a) \geq N^{''}(x, a) \tag{2.36}$$

for all $x \in X$ *and all* $a > 0$, *where*

$$N^{''}(x,a) := min\{N_1^{''}(x, \frac{a(4-\alpha)}{12}, N_3^{''}(x, \frac{a(2-\alpha)}{8}))\}$$

and $N_1^{''}$, $N_3^{''}$ *have been defined in Theorems* 2.1 *and* 2.3, *respectively.*

Proof. Let $f_e(x) = \frac{f(x)+f(-x)}{2}$ for all $x \in X$. Then $f_e(0) = 0$, $f_e(-x) = f_e(x)$ and

$$\begin{aligned}
&N(f_e(2x+y) + f_e(2x-y) - f_e(x+y) - f_e(x-y) - 2f_e(2x) + 2f_e(x), a)\\
&\quad = N(\frac{1}{2}[f(2x+y) + f(2x-y) - f(x+y) - f(x-y) - 2f(2x) + 2f(x)]\\
&\quad + \frac{1}{2}[f(-2x-y) + f(-2x+y) - f(-x-y) - f(-x+y) - 2f(-2x) + 2f(-x)], a)\\
&\quad = N([f(2x+y) + f(2x-y) - f(x+y) - f(x-y) - 2f(2x) + 2f(x)]\\
&\quad + [f(-2x-y) + f(-2x+y) - f(-x-y) - f(-x+y) - 2f(-2x) + 2f(-x)], 2a)\\
&\quad \geq min\{N^{'}(\varphi(x,y),a), N^{'}(\varphi(-x,-y),a)\} \qquad (2.37)
\end{aligned}$$

for all $x, y \in X$ and $a > 0$. Hence, there exists a unique quadratic function $Q : X \to Y$ satisfying

$$N(Q(x) - f_e(x), a) \geq N_1^{''}(x, \frac{a(4-\alpha)}{6} \qquad (2.38)$$

for all $x \in X$ and all $a > 0$. Let $f_o(x) = \frac{f(x)-f(-x)}{2}$ for all $x \in X$. Then $f_o(0) = 0$, $f_o(-x) = -f_o(x)$ and

$$\begin{aligned}
&N(f_o(2x+y) + f_o(2x-y) - f_o(x+y) - f_o(x-y) - 2f_o(2x) + 2f_o(x), a)\\
&\quad \geq min\{N^{'}(\varphi(x,y),a), N^{'}(\varphi(-x,-y),a)\}
\end{aligned}$$

for all $x, y \in X$ and $a > 0$. From Theorem 2.3, there exists a unique additive function $A : X \to Y$ satisfying

$$N(A(x) - f_o(x), a) \geq N_3^{''}(x, \frac{a(2-\alpha)}{4}) \qquad (2.39)$$

for all $x \in X$ and all $a > 0$. Hence (2.36) follows from (2.38) and (2.39). □

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Department of Mathematics, Semnan University, P. O. Box 35195-363, Semnan, Iran and Section of Mathematics and Informatics, Pedagogical Department, National and Capodistrian University of Athens, 4, Agamemnonos St., Aghia Paraskevi, Athens 15342, Greece

E-mail address: madjid.eshaghi@gmail.com, ghobadipour.n@gmail.com, jrassias@primedu.uoa.gr